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Buckling analysis of plain knitted fabric sheets under simple shear in an arbitrary direction

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Abstract

Knitting structures make plain knitted fabric different from woven fabric. With the aid of a micro-constitutive model the buckling of a knitted fabric sheet subjected to simple shear in an arbitrary direction is investigated. The large deformation of the fabric sheet in its critical configuration is considered. The theory of stability for finite deformations is applied to the analysis. All the stress boundary conditions of the knitted fabric sheet are satisfied. An equation for determining the buckling direction angle is derived. It is shown that there are two possible buckling modes: a flexural mode and a barreling mode. The buckling conditions for the two modes are also obtained, respectively. A numerical calculation reveals that only the flexural mode can occur, which agrees with experimental observations.

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1. Introduction

Simulations of draping and buckling/wrinkling of fabric sheet have attracted many researchers' attention because of their great prospect of application in developing "Trial Systems of Apparel" which can model how clothes would appear on models in fashion shows, see, e.g., [Amirbayat and Hearle \(1989\)](#). The draping and buckling of woven fabric have extensively been studied, see, e.g., [Kim \(1991\)](#), [Kang and Yu \(1995\)](#), [Chen and Govindaraj \(1995\)](#), [Zhang and Fu \(2000, 2001\)](#), [Zhang and Xu \(2002a,b\)](#) and [Zhang and Xie \(2003\)](#). These results have paved the way for the development of a trial system of apparel made of woven fabric.

For more details about the recent advances of woven fabric mechanics we refer readers to the review article by [Zhang \(2003a\)](#) and the book by [Zhang \(2003b\)](#).

However, very few papers on simulations of draping and buckling/wrinkling of knitted fabric have appeared in the open literature. As knitting structures are more complicated than woven structures, there does not yet exist a suitable constitutive model for describing the mechanical behavior of knitted fabric. Simple tests

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demonstrate that the mechanical behavior of knitted fabric is very different from that of woven fabric. For example, a plain woven fabric sheet subjected to a tension along warp/weft direction does not buckle (Zhang and Fu, 2001), but a plain knitted fabric sheet under tension along a non-course direction can buckle and the one under tension along a course-direction cannot buckle; see the photos shown of Fig. 1 in Zhang et al. (2004).

The knitting structures composed of interlocking loops of yarns make knitted fabric very different from woven fabric: the former, for instance, is much “softer”, easier to buckle, and hence more suitable for underwear (thus, it is well known that shirts are usually made of woven fabric). In order to characterize its mechanical properties *mathematically*, a micro-constitutive model for plain knitted fabric has been proposed by Zhang et al. (2004). Based on the model the out-of-plane buckling of knitted fabric sheet subjected to a tension along the wale direction has been predicted successfully, see Zhang et al. (2005).

The buckling of a knitted fabric sheet under simple shear in an arbitrary direction is investigated in the present paper. Since the deformation of a knitted fabric sheet at the critical configuration is usually very large, the theory of stability used in the present paper is the well-established theory on incremental deformations; see Ogden (1984), Fu and Ogden (1999), where a small-amplitude buckling mode is superimposed on a large deformation at the critical configuration.

Fabric sheets are very thin, and the theory of thin plates or the theory of thin shells is usually adopted to simulate its deformations due to the simplicity of the theories, although the theories are only an approximation to the three-dimensional theory. In the present analysis, however, we choose to work with the three-dimensional theory of continuum for the buckling analysis of knitted fabric sheets.

The rest of this paper is organized into six sections as follows. After briefly stating the micro-constitutive model of plain knitted fabric in Section 2, the deformation of a knitted fabric sheet under simple shear in an arbitrary direction is described in Section 3. Buckling equations and buckling analysis are provided in Sections 4 and 5, respectively. In Section 6 a numerical illustration is presented. Finally, in Section 7, we summarize our main results and make some suggestions for future research.

2. Micro-constitutive model of plain knitted fabric

Plain knitted fabric is made of interlocking loops of yarns. The direction in which the loops are interlocked is called *wale-direction*; and the direction along which the loops line up in a course is called *course-direction*. A Cartesian coordinate system OWC , in which the W - and C -axis are along the wale- and course-direction respectively as shown in Fig. 1, is usually used in analysis to point out the wale- and the course-direction of a plain knitted fabric sheet. It is assumed that the loops in a typical knitting structure are circles and adjacent loops are tangent to each other at their contact points. The interlocking loops of yarns play two

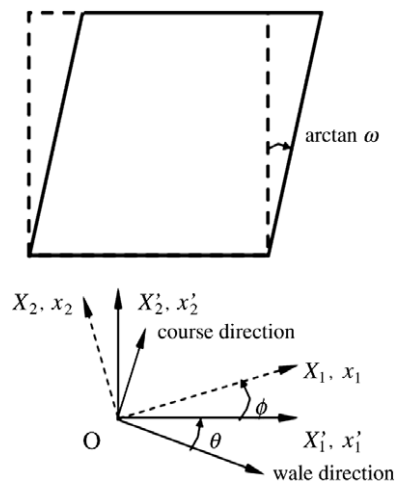


Fig. 1. Coordinate systems for a knitted fabric sheet subjected to simple shear along the X'_1 -axis.

important roles: firstly, the “interlocking” between loops connects loops to form fabric sheets and transfer loading, and secondly, the loop-shaped yarns in knitted fabric can undergo larger extensive deformation than the straight yarns in woven fabric. Tests have shown that the tensile rigidity of knitted fabric is much less than that of woven fabric composed of the same yarns.

Since the second Piola-Kirchhoff stress tensor and the Green strain tensor are work-conjugate and are usually adopted in engineering the micro-constitutive relation for plain knitted fabric is set up between them and has the following form, see Zhang et al. (2004),

$$\begin{Bmatrix} T_w \\ T_c \\ T_{wc} \end{Bmatrix} = \begin{Bmatrix} T_w^0 \\ T_c^0 \\ T_{wc}^0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ T_c^{\text{extra}} \\ 0 \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{21} & B_{22} & 0 \\ 0 & 0 & 2B_{66} \end{bmatrix} \begin{Bmatrix} E_w \\ E_c \\ E_{wc} \end{Bmatrix}, \quad (1)$$

where

$$B_{21} = B'_{21} - \xi(\bar{q}_w)B_{11}, \quad B_{22} = B'_{22} - \xi(\bar{q}_w)B_{12}. \quad (2)$$

Eq. (1) is a modified orthotropic constitutive model, in that the following orthotropic constitutive model is involved

$$\begin{Bmatrix} T_w^0 \\ T_c^0 \\ T_{wc}^0 \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B'_{21} & B'_{22} & 0 \\ 0 & 0 & B_{66} \end{bmatrix} \begin{Bmatrix} E_w \\ E_c \\ E_{wc} \end{Bmatrix}, \quad (3)$$

where

$$B_{11} = \frac{Y_w}{1 - \nu_c \nu_w}, \quad B_{12} = \frac{Y_w \nu_w}{1 - \nu_c \nu_w}, \quad B'_{21} = \frac{Y_c \nu_c}{1 - \nu_c \nu_w}, \quad B'_{22} = \frac{Y_c}{1 - \nu_c \nu_w}, \quad B_{66} = G_{wc}/2 \quad (4)$$

with Y_w and Y_c being the Young moduli in the wale- and course-directions, respectively, ν_w and ν_c the Poisson ratios in wale- and course-directions, respectively, and G_{wc} the shear modulus. The Y_w and Y_c can be determined by tensile tests, see Zhang et al. (2004), and G_{wc} can be determined approximately by KES (Kawabata evaluation system for fabric) shear test, see Hu and Zhang (1997).

The term T_c^{ext} in Eq. (1) represents an extra compressive stress field inside the fabric induced by knitting structures, and has the following representation

$$T_c^{\text{extra}} = -\xi(\bar{q}_w)T_w^0, \quad (5)$$

where $\xi(\bar{q}_w)$ is a nonlinear function of \bar{q}_w , and its detailed representation can be found in Zhang et al. (2004), and \bar{q}_w is defined by

$$\bar{q}_w = \lambda_w T_w, \quad (6)$$

where $\lambda_w = F_{11}^w$, and F_{11}^w is a component of deformation gradient related to the coordinate system *OWC*. When $\bar{q}_w/B_{66} = 0.1, 0.2, \dots, 1.0$ the values of $\xi(\bar{q})$ for $R = 1$ mm and $\eta = 1$ are listed in Table 1, where R is the radius of loops in the reference configuration, and η is a material constant of yarns which appears in the mechanical model of yarn

$$d = De^{-\eta p} \quad (7)$$

with D and d being the undeformed and deformed diameters of yarn, respectively, p being the pressure loaded on yarn in its radius directions, and e being the Euler constant.

Table 1
The values of $\xi(\bar{q})$ varying with \bar{q}_w/B_{66}

\bar{q}_w/B_{66}	0.1	0.2	0.3	0.4	0.5
$\xi(\bar{q})$	0.1952	0.1891	0.1840	0.1797	0.1760
\bar{q}_w/B_{66}	0.6	0.7	0.8	0.9	1.0
$\xi(\bar{q})$	0.1728	0.1699	0.1675	0.1653	0.1636

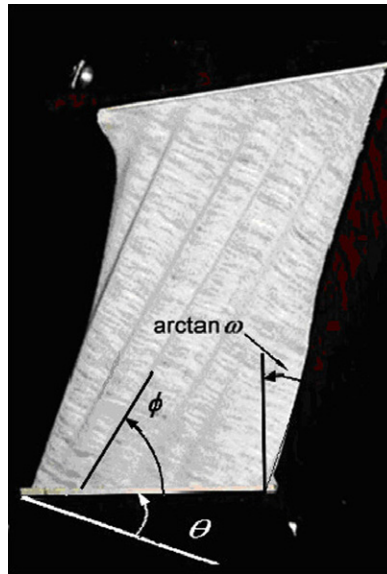


Fig. 2. Buckled knitted fabric specimen under quasi-simple shear along the direction making an angle θ with respect to the wale direction.

It is worth noting that the extra compressive stress field T_c^{ext} exists only inside knitted fabric and does not take part in the satisfaction of stress boundary conditions of knitted fabric sheets. The stress boundary conditions take the form

$$T_{AB}^0 N_B = \bar{T}_A, \quad A, B = 1, 2, \quad (8)$$

where \bar{T}_A is the traction applied to the boundary which has unit outward normal N_A in the reference configuration.

Based on the above micro-constitutive model the out-of-plane buckling of knitted fabric sheets subjected to tension along the wale direction has been predicted successfully, see Zhang et al. (2005). In the following sections we will investigate the buckling of knitted fabric sheets subjected to simple shear along the direction making an angle, say θ , with respect to the wale direction as shown in Fig. 1, where ω is the amount of shear.

A post-buckled knitted fabric specimen shown in Fig. 2 is subjected to a quasi-simple shear (for the reasons of “quasi-simple shear” to be mentioned at the end of Section 3) along the direction making an angle of θ with respect to the wale direction. Similar buckling patterns can also be observed in the KES shear tests on knitted fabric sheets. In this paper we shall focus our attention on predicting the possible buckling modes and the corresponding buckling conditions. From the photo shown in Fig. 2, the direction of buckling wave trough does not coincide with the loading direction. We denote by ϕ the angle between the direction of buckling wave trough and the shear direction (see Fig. 2). The angle ϕ is referred to as the *buckling direction angle* and should be determined in analysis.

3. Homogeneous deformation of knitted fabric sheet under simple shear

We consider a piece of knitted fabric as shown in Fig. 1. We set up another Cartesian coordinates system $OX'_1X'_2X'_3$: the X'_3 -axis is normal to the mid-plane of fabric sheet (thus the X'_3 -axis cannot be plotted in Fig. 1), the X'_1 - and X'_2 -axes are obtained by rotating the W - and C -axes about the X'_3 -axis counter-clock-wise by an angle θ . The thickness of fabric sheet is denoted by $2h$.

We assume that the boundary surfaces $X'_3 = \pm h$ of the knitted fabric sheet are traction free and an application of simple shear along the X'_1 -direction carries the initial configuration of the sheet, denoted by B_0 and plotted in Fig. 1 in dashed lines, into another homogeneous equilibrium configuration, denoted by B_e and plotted in solid lines. Relative to the same Cartesian coordinate system $OX'_1X'_2X'_3$ a material particle which has coordinates (X'_i) in B_0 has coordinates (x'_i) in B_e .

The homogeneous deformation of the fabric sheet under simple shear can be represented by

$$x'_1 = X'_1 + \omega X'_2, \quad x'_2 = X'_2, \quad x'_3 = X'_3, \quad (9)$$

where ω is the amount of the shear. It is clear that the deformation gradient tensor has the form

$$\bar{\mathbf{F}}' = \begin{bmatrix} 1 & \omega & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (10)$$

where the bar over \mathbf{F}' indicates that the deformation gradient tensor arises from $B_0 \rightarrow B_e$, and the superscript “prime” indicates that the components of the tensor are related to the coordinate system $OX'_1X'_2X'_3$.

Since the boundary surfaces $X'_3 = \pm h$ are traction free it follows that the following components of stress tensor inside the fabric sheet are zero:

$$\bar{T}'_{33} = \bar{T}'_{23} = \bar{T}'_{31} = 0. \quad (11)$$

By the micro-constitutive Eq. (1) and the transformation rule of tensors the other nonzero components of the stress tensor $\bar{\mathbf{T}}'$ inside the knitted fabric sheet should be as follows

$$\begin{Bmatrix} \bar{T}'_{11} \\ \bar{T}'_{22} \\ \bar{T}'_{12} \end{Bmatrix} = \mathbf{R}' \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{21} & B_{22} & 0 \\ 0 & 0 & 2B_{66} \end{bmatrix} \mathbf{R}'^{-1} \begin{Bmatrix} \bar{E}'_{11} \\ \bar{E}'_{22} \\ \bar{E}'_{12} \end{Bmatrix}, \quad (12)$$

where the superscript “ -1 ” signifies the inverse of a matrix, and \mathbf{R}' is the transformation tensor corresponding to $OWC \rightarrow OX'_1X'_2$ and its matrix form is as follows

$$\mathbf{R}' = \begin{bmatrix} \cos^2 \theta & \sin^2 \theta & 2 \cos \theta \sin \theta \\ \sin^2 \theta & \cos^2 \theta & -2 \cos \theta \sin \theta \\ -\cos \theta \sin \theta & \cos \theta \sin \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix}. \quad (13)$$

With the aid of $\bar{\mathbf{E}}' = \frac{1}{2}(\bar{\mathbf{F}}'^T \bar{\mathbf{F}}' - \mathbf{I})$, where \mathbf{I} is the identity matrix and the superscript ‘T’ signifies transpose of a tensor, for a given ω the nonzero components of strain tensor $\bar{\mathbf{E}}'$ are

$$\bar{E}'_{12} = \bar{E}'_{21} = \frac{1}{2}\omega, \quad \bar{E}'_{22} = \frac{1}{2}\omega^2. \quad (14)$$

With the aid of the transformation rule of tensors we obtain

$$\bar{\mathbf{F}}^w = \mathbf{R}'^{-1} \bar{\mathbf{F}}', \quad (15)$$

where $\bar{\mathbf{F}}^w$ is the deformation gradient tensor arising from $B_0 \rightarrow B_e$ and related to the coordinate system OWC . We have

$$\lambda_w = \bar{F}^w_{11}. \quad (16)$$

Again, with the aid of the transformation rule of tensors, we also obtain

$$\begin{Bmatrix} \bar{E}_w \\ \bar{E}_c \\ \bar{E}_{wc} \end{Bmatrix} = \mathbf{R}'^{-1} \begin{Bmatrix} \bar{E}'_{11} \\ \bar{E}'_{22} \\ \bar{E}'_{12} \end{Bmatrix}. \quad (17)$$

With the aid of Eq. (2), substituting Eq. (14) into Eq. (17) and then the result into Eq. (1) gives the homogeneous deformation field in the knitted fabric sheet under simple shear as follows

$$\begin{Bmatrix} \bar{T}_w \\ \bar{T}_c \\ \bar{T}_{wc} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B'_{21} - \xi(\lambda_w \bar{T}_w) B_{11} & B'_{22} - \xi(\lambda_w \bar{T}_w) B_{12} & 0 \\ 0 & 0 & 2B_{66} \end{bmatrix} \mathbf{R}'^{-1} \begin{Bmatrix} 0 \\ \frac{1}{2}\omega^2 \\ \frac{1}{2}\omega \end{Bmatrix}. \quad (18)$$

For a given ω , λ_w can be determined through Eqs. (16), (15) and (10), and then \bar{T}_w , \bar{T}_c and \bar{T}_{wc} can be evaluated from Eq. (18).

By Eq. (8), the tractions on marginal boundaries of a finite fabric sheet subjected to a simple shear should be

$$\begin{Bmatrix} \bar{T}'_{11} \\ \bar{T}'_{22} \\ \bar{T}'_{12} \end{Bmatrix} = \mathbf{R}' \begin{bmatrix} B_{11} & B_{12} & 0 \\ B'_{21} & B'_{22} & 0 \\ 0 & 0 & 2B_{66} \end{bmatrix} \mathbf{R}'^{-1} \begin{Bmatrix} \bar{E}'_{11} \\ \bar{E}'_{22} \\ \bar{E}'_{12} \end{Bmatrix}. \quad (19)$$

By Eq. (14), \bar{E}'_{22} and \bar{E}'_{12} are not equal to zero in general, so that \bar{T}'_{11} , \bar{T}'_{22} and \bar{T}'_{12} on the marginal boundaries of the knitted fabric sheet under simple shear are not zero in general. Thus, the fabric specimen shown in Fig. 2, whose partial marginal boundaries are traction free, is not under simple shear, and we term the “shear” in Fig. 2 “quasi-simple shear”.

In the remaining sections the buckling of the knitted fabric sheet under simple shear will be predicted analytically. As the deformation of a knitted fabric sheet at the critical configuration is usually large, the theory of stability for finite deformations, see Ogden (1984) and Fu and Ogden (1999), will be applied.

4. Governing equations

We assume that under the same boundary conditions another non-homogeneously deformed configuration, called *buckled configuration* and denoted by B_t , also exists. In the remaining sections we choose another coordinate system $OX_1X_2X_3$ which is obtained by rotating the X'_1 - and X'_2 -axes about the X'_3 -axis counter-clockwise by an angle ϕ and the X_3 -axis coincides with the X'_3 -axis as shown in Fig. 1. We will see that ϕ is just the angle between the X'_1 -axis and the direction of wave trough of the buckled fabric sheet. Thus, ϕ is referred to as the *buckling direction angle*. From the photo in Fig. 2 the direction of wave trough of the buckled fabric sheet does not coincide with the shear direction (X'_1 -axis). The value of the angle ϕ needs to be determined in analysis.

Relative to the same coordinate system $OX_1X_2X_3$ a material particle which has coordinates (X_A) in B_0 has coordinates (x_i) in B_e and has (x'_i) in B_t . We write

$$x'_i = x_i(X_A) + u_i(x_j), \quad (20)$$

where $u_i(x_j)$ is a small amplitude displacement associated with the deformation $B_e \rightarrow B_t$.

It is convenient to introduce

$$\chi_{ij} = \bar{J}^{-1}(\pi_{iA} - \pi_{iA})\bar{F}_{jA}, \quad (21)$$

where π_{iA} are the components of the first Piola-Kirchhoff stress tensor associated with the deformation $B_0 \rightarrow B_t$, π_{iA} the components of the first Piola-Kirchhoff stress tensor and \bar{F}_{jA} the components of deformation gradient tensor associated with the deformation $B_0 \rightarrow B_e$, and $\bar{J} = \det \bar{F}$. The π_{iA} , \bar{F}_{jA} and π_{iA} (without superscript ‘prime’) are all related to the coordinate system $OX_1X_2X_3$.

With the aid of Eq. (21) the equations of equilibrium in the absence of body forces can be written as

$$\chi_{ij,j} = 0, \quad (22)$$

and the dead-load boundary conditions can be written as

$$\chi_{ij}n_j = 0, \quad (23)$$

where n_j ($j = 1, 2, 3$) are the components of the unit out-normal vector to the boundary of the fabric sheet in B_e .

It is usually assumed that the mid-plane of fabric sheet is a plane of elastic symmetry. Denoting the components of the second Piola-Kirchhoff stress tensor and the Green strain tensor relative to the coordinate system $OX_1X_2X_3$ by T_{ij} ($i, j = 1, 2, 3$) and E_{ij} ($i, j = 1, 2, 3$), respectively, we write the constitutive equations for the knitted fabric as

$$\begin{Bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{12} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 & 0 & 0 & 2D_{16} \\ D_{21} & D_{22} & 0 & 0 & 0 & 2D_{26} \\ 0 & 0 & D_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2D_{55} & 0 \\ D_{61} & D_{62} & 0 & 0 & 0 & 2D_{66} \end{bmatrix} \begin{Bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ E_{23} \\ E_{31} \\ E_{12} \end{Bmatrix}, \quad (24)$$

where

$$\begin{bmatrix} D_{11} & D_{12} & 2D_{16} \\ D_{21} & D_{22} & 2D_{26} \\ D_{61} & D_{62} & 2D_{66} \end{bmatrix} = \mathbf{R} \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{21} & B_{22} & 0 \\ 0 & 0 & 2B_{66} \end{bmatrix} \mathbf{R}^{-1}, \quad (25)$$

with \mathbf{R} being the coordinate transformation matrix corresponding to $OWC \rightarrow OX_1X_2$ which can be obtained from \mathbf{R}' by replacing θ by $(\theta + \phi)$.

Eq. (24) can be rewritten in the tensor form

$$T_{ij} = L_{ijkl}E_{kl}. \quad (26)$$

By comparing Eq. (24) with Eq. (26) the non-zero components of L_{ijkl} can be obtained and are given in Appendix A.

The representation of the linearized χ_{ij} in terms of $u_{k,l}$ is as follows (Zhang and Fu, 2001)

$$\chi_{ij} = A_{jilk}u_{k,l}, \quad (27)$$

where (A_{jilk}) is the fourth-order instantaneous elastic modulus tensor and has the following expression (Zhang and Fu, 2001)

$$A_{jilk} = \bar{J}^{-1} \bar{F}_{j\alpha} \bar{F}_{l\beta} L_{\alpha\beta\gamma\epsilon} \bar{F}_{i\gamma} \bar{F}_{ke} + \bar{J}^{-1} \bar{F}_{i\alpha} \bar{F}_{k\beta} \bar{T}_{\alpha\beta} \delta_{jl}, \quad (28)$$

where δ_{jl} is the Kronecker delta.

5. Buckling analysis

We now look for a buckling solution of the form

$$u_1 = 0, \quad u_j = H_j(kx_3)e^{ikx_2} + \text{C.C.}, \quad j = 2, 3, \quad (29)$$

where $i = \sqrt{-1}$, k is the mode number, and C.C. denotes the complex conjugate of the preceding term. Substituting Eq. (29) into Eq. (27) and then the result into Eq. (22) gives

$$C_{11}H_2(kx_3) + C_{12}[iH'_3(kx_3) + H''_2(kx_3)] = 0, \quad (30)$$

$$C_{21}H_3(kx_3) + iC_{22}H'_2(kx_3) + D_{33}H''_3(kx_3) = 0, \quad (31)$$

$$C_{31}H_2(kx_3) + C_{32}[iH'_3(kx_3) + H''_2(kx_3)] = 0, \quad (32)$$

where the coefficients C_{ij} ($i = 1, 3; j = 1, 2$) are polynomials that involve \bar{F}_{ij} , \bar{T}_{ij} and D_{ij} . The \bar{F}_{ij} and \bar{T}_{ij} are related to the loading parameter ω and the buckling direction angle ϕ . Thus the C_{ij} are related to ω and ϕ too. The expressions of the C_{ij} can be found in Appendix B.

Thus, two unknown functions $H_2(kx_3)$ and $H_3(kx_3)$, an unknown variable ϕ and a loading parameter ω are involved in Eqs. (30)–(32).

If we let

$$C_{11}C_{32} - C_{12}C_{31} = 0, \quad (33)$$

it follows that Eqs. (30) and (32) are the same in the sense that they have the same solutions for $H_2(kx_3)$ and $H_3(kx_3)$. Eq. (33) is simply the equation to determine the buckling direction angle ϕ for a given value of ω .

Solving the two second-order differential Eqs. (30) and (31) for H_2 and H_3 yields

$$-iH_2(kx_3) = A_2 \sinh(p_1 kx_3) + A_4 \sinh(p_2 kx_3) + A_1 \cosh(p_1 kx_3) + A_3 \cosh(p_2 kx_3), \quad (34)$$

$$H_3(kx_3) = q_1[A_2 \cosh(p_1 kx_3) + A_1 \sinh(p_1 kx_3)] + q_2[A_4 \cosh(p_2 kx_3) + A_3 \sinh(p_2 kx_3)], \quad (35)$$

where A_j ($j = 1, 2, 3, 4$) are constants of integrations that are determined by imposing the boundary conditions (23), and the expressions of q_j ($j = 1, 2$) and p_j ($j = 1, 2$) can be found in Appendix C.

Substituting Eqs. (34) and (35) into Eq. (29) yields the solutions for u_2 and u_3 , and then substituting them into the boundary conditions (23) at $x_3 = \pm h$ for the free indices $i = 2, 3$ gives

$$b_1(h) = 0, \quad b_1(-h) = 0, \quad b_2(h) = 0, \quad b_2(-h) = 0, \quad (36)$$

where the expressions of b_1 and b_2 can be found in Appendix D.

Substituting the solutions for u_2 and u_3 into the boundary conditions (23) at $x_3 = \pm h$ for the free index $i = 1$ gives the same equations as the first two of Eq. (36).

Simplifying the first and second equations of Eq. (36) gives

$$A_2 p_2 \cosh(hkp_1) + A_4 p_1 \cosh(hkp_2) = 0, \quad (37)$$

$$A_1 p_2 \sinh(hkp_1) + A_3 p_1 \sinh(hkp_2) = 0. \quad (38)$$

And simplifying the third and fourth ones of Eq. (36) gives

$$A_2(C_{11} + C_{12}p_1^2) \sinh(hkp_1) + A_4(C_{11} + C_{12}p_2^2) \sinh(hkp_2) = 0, \quad (39)$$

$$A_1(C_{11} + C_{12}p_1^2) \cosh(hkp_1) + A_3(C_{11} + C_{12}p_2^2) \cosh(hkp_2) = 0. \quad (40)$$

Eqs. (37) and (39) are two linearly homogenous equations for A_2 and A_4 which have non-trivial solutions (in other words, the buckling configuration exists) only if the determinant of their coefficient matrix vanishes, namely

$$(C_{11} + C_{12}p_2^2)p_2 \cosh(hkp_1) \sinh(hkp_2) - (C_{11} + C_{12}p_1^2)p_1 \sinh(hkp_1) \cosh(hkp_2) = 0. \quad (41)$$

By analogy with the analysis above, Eqs. (38) and (40) have non-trivial solutions for A_1 and A_3 only if the following equation is satisfied:

$$(C_{11} + C_{12}p_2^2)p_2 \sinh(hkp_1) \cosh(hkp_2) - (C_{11} + C_{12}p_1^2)p_1 \cosh(hkp_1) \sinh(hkp_2) = 0. \quad (42)$$

It is not possible that Eqs. (41) and (42) are satisfied simultaneously. When Eq. (41) is satisfied Eqs. (37) and (39) have no-zero solutions for A_2 and A_4 , and Eq. (38) and (40) have only the trivial solution for A_1 and A_3 . In this case, by Eqs. (29) and (35), the corresponding u_3 is an even function of x_3 and the buckling solution is referred to as *flexural mode*. The buckling model of the knitted fabric sheet shown in Fig. 2 is just a flexural one.

In the event that Eq. (42) is satisfied, there may be no-zero solutions for A_1 and A_3 . In this case the corresponding u_3 is an odd function of x_3 and the buckling solution will be referred to as *extensional mode* or *barreling mode*.

In Section 6 a numerical illustration will show that the barreling mode does not occur for the knitted fabric sheet under simple shear, which agrees with the fact that no barreling mode of knitted fabric sheets has ever been observed experimentally.

6. Illustration

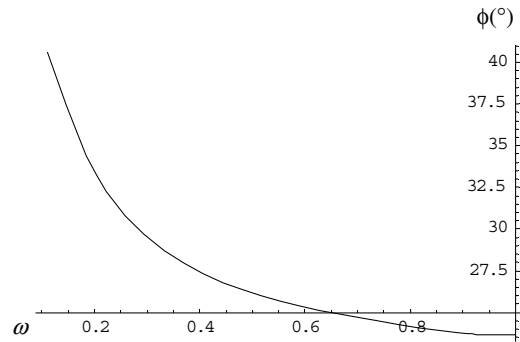
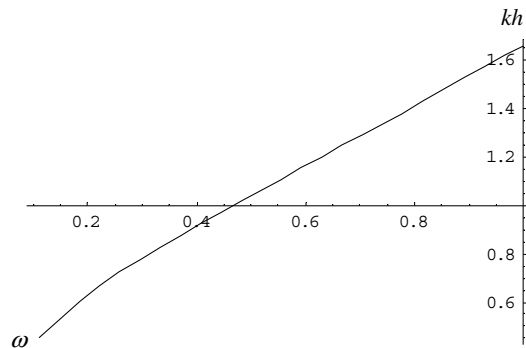
As an illustration we assume that

$$\frac{B_{11}}{B_{66}} = \frac{B'_{22}}{B_{66}} = 2, \quad \frac{D_{33}}{B_{66}} = 3, \quad \frac{B_{12}}{B_{66}} = \frac{B'_{21}}{B_{66}} = 0.3, \quad \frac{D_{44}}{B_{66}} = \frac{D_{55}}{B_{66}} = 2, \quad (43)$$

$$R = 1 \text{ mm}, \quad \eta = 1, \quad (44)$$

For a fixed value of θ , the values of $\omega = 0.1, 0.2, \dots, 0.8$ are assigned. For every assigned value of ω , the value of $\xi(\bar{q}_w)$ can be determined from Table 1 (interpolations are usually needed). With the aid of the software packet Mathematica, solving Eq. (33) gives the value of ϕ . The curve of ϕ against ω for $\theta = 30^\circ$ is plotted in Fig. 3.

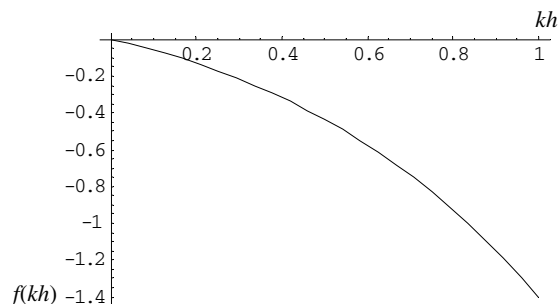
For every pair of values of ω and ϕ , solving Eq. (41) gives the value of kh for flexural model. When $\theta = 30^\circ$ the curve of kh against ω is plotted in Fig. 4. The curve of kh against ω is referred to as the *buckling condition*

Fig. 3. The curve of ϕ against ω for $\theta = 30^\circ$.Fig. 4. The buckling condition curve for flexural mode when $\theta = 30^\circ$.

curve for flexural model. The value of ω corresponding to the minimum value of kh is the critical amount of simple shear. The minimum value of kh should be determined from restraints on displacements at the boundaries of the knitted fabric sheet. The buckling analysis in the present paper is quite general and no restraints on displacement at the boundaries are imposed. Only stress boundary conditions are concerned in this analysis.

If restraints on displacement at the boundaries are introduced the critical value of ω and the buckling direction angle ϕ will be determined. For example, if the restraints on displacement at the boundaries permit a buckling mode such as $kh = a$, substituting $kh = a$ into Eqs. (33) and (41) and then solving them simultaneously give just the critical value of ω and the buckling direction angle ϕ .

The upper and lower boundaries of the knitted fabric sheet shown in Fig. 2 are fixed and do not permit any buckling at the boundaries, but buckling still occurs. The experiment demonstrates that buckling of knitted fabric sheets under quasi-shear is an inherent behavior. It is difficult to simulate buckling of the knitted fabric

Fig. 5. Only trivial solution $kh = 0$ exists for the barreling mode.

specimen shown in Fig. 2 analytically because its deformation at the critical configuration is not homogeneous, but numerical methods such as the Finite Element Methods are available.

For every pair of values of ω and ϕ obtained from the curve of ϕ against ω in Fig. 3, solving Eq. (42) gives only the trivial solution for kh , i.e. there is not any nonzero solution for kh . For example, for $\omega = 0.8$ the curve of the function on the left of Eq. (42), denoted by $f(kh)$, against kh is plotted in Fig. 5. It clearly shows that Eq. (42) has only the zero solution for kh . This means that the barreling mode does not occur and only the flexural mode can occur when the knitted fabric sheet is subjected to simple shear. This agrees with experimental observations: no barreling modes have ever been observed.

7. Conclusions

Experiments show that knitted fabric sheets can buckle easily. Based on a micro-constitutive model for plain knitted fabric the buckling of a knitted fabric sheet under simple shear is investigated analytically. For a fixed shear direction making an angle θ with the wale direction and the permitted minimum value of buckling mode kh on margin boundaries, the critical value of shear amount ω and the buckling direction angle ϕ can be determined. It is also verified that only the flexible mode can occur for a knitted fabric sheet under simple shear and the barreling mode cannot occur, which agrees with experimental observations.

Since the analytical results given in this paper are rigorous and accurate they can also be utilized as a benchmark for a fully numerical scheme such as one based on Finite Element Methods.

The analytical method provided in the present paper is suitable for a fabric sheet whose deformation at the critical configuration is homogenous. For situations where the deformation at the critical configuration is inhomogeneous, such as the knitted fabric specimen shown in Fig. 2, one would have to resort to numerical methods.

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Appendix A

$$\begin{aligned} L_{1111} &= D_{11}, & L_{2222} &= D_{22}, & L_{3333} &= D_{33}, & L_{1122} &= D_{12}, & L_{2211} &= D_{21}, \\ L_{1112} &= L_{1121} = D_{16}, & L_{2212} &= L_{2221} = D_{26}, \\ L_{1211} &= L_{2111} = D_{61}, & L_{1222} &= L_{2122} = D_{62}, \\ L_{2323} &= L_{2332} = L_{3223} = L_{3232} = D_{44}, & L_{3131} &= L_{3113} = L_{1331} = L_{1313} = D_{55}, \\ L_{1212} &= L_{1221} = L_{2112} = L_{2121} = D_{66} \end{aligned}$$

Appendix B

$$\begin{aligned} C_{11} &= -D_{11}\bar{F}_{21}^4 - (2D_{16} + 2D_{61})\bar{F}_{21}^3\bar{F}_{22} - (D_{12} + D_{21} + 4D_{66})\bar{F}_{21}^2\bar{F}_{22}^2 \\ &\quad - (2D_{26} + 2D_{62})\bar{F}_{21}\bar{F}_{22}^3 - D_{22}\bar{F}_{22}^4 - \bar{T}_{11}\bar{F}_{21}^2 - 2\bar{T}_{12}\bar{F}_{21}\bar{F}_{22} - \bar{T}_{22}\bar{F}_{22}^2 \\ C_{21} &= -D_{55}\bar{F}_{21}^2 - D_{44}\bar{F}_{22}^2 - \bar{T}_{11}\bar{F}_{21}^2 - 2\bar{T}_{12}\bar{F}_{21}\bar{F}_{22} - \bar{T}_{22}\bar{F}_{22}^2 \\ C_{12} &= C_{22} = D_{55}\bar{F}_{21}^2 + D_{44}\bar{F}_{22}^2 \\ C_{31} &= -D_{11}\bar{F}_{11}\bar{F}_{21}^3 - D_{61}\bar{F}_{21}^3\bar{F}_{12} - (D_{61} + D_{16})\bar{F}_{11}\bar{F}_{21}^2\bar{F}_{22} - (D_{21} + 2D_{66})\bar{F}_{12}\bar{F}_{21}^2\bar{F}_{22} \\ &\quad - (D_{12} + 2D_{66})\bar{F}_{11}\bar{F}_{21}\bar{F}_{22}^2 - (D_{26} + D_{62})\bar{F}_{12}\bar{F}_{21}\bar{F}_{22}^2 - D_{62}\bar{F}_{11}\bar{F}_{22}^3 - D_{22}\bar{F}_{12}\bar{F}_{22}^3 \\ C_{32} &= D_{55}\bar{F}_{11}\bar{F}_{21} + D_{44}\bar{F}_{12}\bar{F}_{22} \end{aligned}$$

Appendix C

$$q_j = -\frac{C_{11} + C_{12}p_j^2}{C_{12}p_j}, \quad j = 1, 2$$

where the p_1 and p_2 are two of four roots of

$$C_{12}C_{22}p^2 + (C_{11} + C_{12}p^2)(C_{21} + D_{33}p^2) = 0$$

that are given by

$$p_1 = \sqrt{m + \sqrt{n}}, \quad p_2 = \sqrt{m - \sqrt{n}}$$

where

$$m = -\frac{1}{2C_{12}D_{33}}(C_{12}C_{21} + C_{12}C_{22} + C_{11}D_{33}), \quad n = -\frac{\sqrt{r}}{2C_{12}D_{33}}$$

$$r = (C_{12}C_{21} + C_{12}C_{22} + C_{11}D_{33})^2 - 4C_{11}C_{12}C_{21}D_{33}$$

Appendix D

$$b_1(h) = (A_4 - A_3)p_1 \cosh(hkp_1) + (A_2 - A_1)p_2 \cosh(hkp_2)$$

$$+ (A_4 - A_3)p_1 \sinh(hkp_1) + (A_2 - A_1)p_2 \sinh(hkp_2)$$

$$+ (A_1 + A_2)p_2 \{ \cosh[hk(2p_1 + p_2)] + \sinh[hk(2p_1 + p_2)] \}$$

$$+ (A_3 + A_4)p_1 \{ \cosh[hk(p_1 + 2p_2)] + \sinh[hk(p_1 + 2p_2)] \}$$

$$b_2(h) = A_1 \cosh(hkp_1)(C_{11} + C_{12}p_1^2) + A_3 \cosh(hkp_2)(C_{11} + C_{12}p_2^2)$$

$$+ A_2 \sinh(hkp_1)(C_{11} + C_{12}p_1^2) + A_4 \sinh(hkp_2)(C_{11} + C_{12}p_2^2)$$

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